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TECHNICAL NOTES

Analysis of hyperbolic heat conduction in a semi-infinite slab with surface convection

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1. INTRODUCTION

Importance of convection

Recent experimental studies [1, 2] provide evidence of non-Fourier heat conduction in some materials with "non-homogeneous inner structures" including porous solids (e.g. sand) and processed meat. These studies propose conduction occurs by thermal wave propagation with sharp wave-fronts separating heated and unheated zones. The non-homogeneous structures apparently induce waves by delaying the response between heat flux and temperature gradient. For example, the delay may represent time needed to accumulate energy for significant heat transfer between structural elements [2]. In comparison, the classical model of Fourier's law permits heat flux to respond immediately to changes in temperature gradient. Consequently, the law does not lead to this type of thermal wave behavior.

The non-Fourier "hyperbolic heat equation," described shortly, appears applicable to some non-homogeneous materials. The study with processed meat [1], for instance, shows good agreement between measured temperatures and predictions using this equation. Further, the study suggests hyperbolic conduction may occur in human tissue as well as processed meat. Although the study does not involve convection heat transfer between meat samples and their surroundings, convection should be important in many applications involving non-homogeneous materials. For example, medical applications often involve convection, as with tissue burns caused by hot liquids [3].

It is important to note non-Fourier conduction is usually associated with "microscale" applications involving very small time and length scales, such as sub-picosecond ($< 10^{-12}$ s) heating of silicon thin films ($< 1\ \mu$) during integrated circuit fabrication. Convection may not be important in microscale applications because of insufficient time for development of fluid motion. However, the experiments [1, 2] with non-homogeneous materials involve "macroscale" non-Fourier conduction since length and time scales are relatively large. Hence, these larger scales can support convection. An expression for estimating convection conditions under which hyperbolic conduction is important is given with the results.

Objective

One analytical study [4] examines hyperbolic conduction in an infinite cylinder with internal Joule heating and convection heat exchange with a surrounding fluid. However, no analytical solution is available that isolates the effect of convection. Thus, the objective here is to obtain this solution for a semi-infinite slab with surface convection for the cases of heating and cooling of the slab. This solution would be convenient for making temperature calculations, correlating future experimental data involving convection and verifying numerical solutions. Also, the solution applies to other geometries under proper conditions. For example, it can apply to a finite slab, as described with the results.

Equations of hyperbolic conduction

Hyperbolic conduction stems from the Cattaneo–Vernotte model relating heat flux and temperature gradient:

$$q+\tau\frac{\partial q}{\partial t} = -k\frac{\partial T}{\partial x} \quad (1)$$

written here for conduction in the x-direction. In equation (1), the thermal relaxation time τ is an approximate measure of the time-delay in heat flux achieving the Fourier's law value after a change in temperature gradient. Measured values for τ are approximately 15 and 20 s for processed meat [1] and sand [2], respectively. The thermal wave speed c is related to relaxation time by $\tau = \alpha/c^2$.

Equation (1) is more general than Fourier's law since setting $\tau = 0$ (immediate response) reduces it to the law. Alternatively, $\tau = 0$ corresponds to $c \to \infty$, implying thermal waves propagate at an infinite velocity for Fourier conduction. Also, equation (1) reduces to Fourier's law for steady-state conditions ($\partial q/\partial t = 0$) even for $\tau \neq 0$.

Combining equation (1) with the statement of energy conservation for a differential control volume gives the hyperbolic heat equation. For constant properties and τ this equation is

$$\tau\frac{\partial^2 T}{\partial t^2}+\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2} \quad (2)$$

which is mathematically classified as hyperbolic. For brevity here, the equation does not include effects of internal energy generation or absorption. In equation (2), $\partial^2 T/\partial t^2$ represents wave propagation of heat and $\partial T/\partial t$ accounts for wave damping (decay). The equation reduces to the Fourier heat equation for $\tau = 0$ or steady-state conditions.

Solutions to hyperbolic conduction problems are non-Fourier thermal waves damped by heat diffusion. In contrast, Fourier solutions show conduction only by diffusion. Further, solutions to hyperbolic problems converge to corresponding Fourier solutions for sufficiently large times after thermal disturbances (e.g. imposition of heat fluxes) because of wave damping. References [5–10] review the history behind equations (1) and (2) and solutions already available for hyperbolic problems.

NOMENCLATURE

c	thermal wave speed
h	heat transfer coefficient
H	unit step function
I	integral defined by equation (7a) or (7b)
k	thermal conductivity
p	Laplace transform parameter
q	heat flux
Q	dimensionless heat flux, $= q/[h(T_\infty - T_0)]$
t	time
T	transient temperature of slab
T_0	initial temperature of slab
T_∞	constant fluid temperature outside of convection boundary layer
u	integration variable in equation (7a)
x	location inside slab
z	integration variable in equation (7b).

Greek symbols

α	thermal diffusivity
ε	dimensionless thermal relaxation time $= \tau\alpha h^2/k^2$
η	dimensionless location inside slab $= xh/k$
θ	dimensionless temperature of slab $= (T-T_0)/(T_\infty - T_0)$
ξ	dimenionless time $= t\alpha h^2/k^2$
τ	thermal relaxation time.

Subscripts

step	value corresponding to step change
wave	value at location of wave-front.

2. ANALYSIS

Formulation

The semi-infinite slab occupies the half-space $x \geqslant 0$ and is initially at temperature T_0. At time $t = 0^+$ the surface along $x = 0$ is exposed to a hot or cold fluid at temperature T_∞. Then, the fluid exchanges heat by convection with the slab, where $T_\infty > T_0$ and $T_\infty < T_0$ correspond to heating and cooling of the slab, respectively. The heat transfer coefficient h, the value of T_∞ outside the convection boundary layer and τ and α for the slab are constant. Also, there is no energy generation or absorption within the slab.

Equation (2) governs $T(x, t)$ for the slab with initial and boundary conditions:

$$T(x,0) = T_0 \tag{3a}$$

$$\frac{\partial T(x,0)}{\partial t} = 0 \tag{3b}$$

$$T(x \to \infty, t) = T_0 \tag{3c}$$

$$q(0,t) = h[T_\infty - T(0,t)]. \tag{3d}$$

Equation (3b) corresponds to an initial heat flux of zero:

$$q(x,0) = 0 \tag{3e}$$

since non-zero heating of the slab for $t < 0$ would cause temperature to already be changing at $t = 0$ [11]. Finally, equation (3d) gives q from convection at $x = 0$. This q must be inserted into equation (1) to specify conduction into or out of the slab at $x = 0$.

Solution

The problem given by equations (1), (3e), (2) and (3a)–(3d) is placed in a more convenient form by introducing the dimensionless time ξ and location η, along with dimensionless temperature $\theta(\eta, \xi)$, relaxation time ε and heat flux Q. The value of θ varies from 0 to 1 for both heating and cooling of the slab.

Next, using the Laplace transform (with parameter p) to eliminate ξ, then solving for the transformed temperature $\bar{\theta}(\eta, p)$ gives

$$\bar{\theta} = \frac{(1+\varepsilon p)}{p} \frac{\exp\{-\eta[p(1+\varepsilon p)]^{1/2}\}}{\{(1+\varepsilon p)+[p(1+\varepsilon p)]^{1/2}\}}. \tag{4}$$

Equation (4) reduces to the corresponding Fourier expression for $\varepsilon = 0$ ($\tau = 0$) or sufficiently large time ($p \ll 1$). This "large time" behavior reflects convergence of the hyperbolic solution to the Fourier solution as the wave decays.

Pausing here, in the analysis, reveals two key features of the wave behavior. First, rewriting equation (4) for sufficiently small time ($p \gg 1$), then inverting the resulting expression gives

$$\theta = \frac{H(\xi - \eta\varepsilon^{1/2})}{(1+\varepsilon^{-1/2})} \tag{5}$$

where H is the unit step function. This step function identifies the thermal wave separating heated and unheated zones in the slab. Specifically, $\theta = 0$ when $\eta > \xi/\varepsilon^{1/2}$ for the unheated zone ahead of the wave. In contrast, $\theta > 0$ when $\eta < \xi/\varepsilon^{1/2}$ for the heated zone behind the wave. Thus, the wave location is $\eta_{\text{wave}} = \xi/\varepsilon^{1/2}$. In fact, reverting to original variables shows $x_{\text{wave}} = ct$, as expected.

The second key feature of wave behavior is the step change in surface temperature that occurs when the slab is exposed to the convection heat flux. Setting $\eta = 0$ and $\xi = 0^+$ in equation (5) shows this step change to be

$$\theta_{\text{step}} = 1/(1+\varepsilon^{-1/2}). \tag{6}$$

This step change is a consequence of delayed conduction into (or out of) the slab, as discussed with the results. As expected, equation (6) shows no step change in surface temperature for Fourier conduction ($\varepsilon = 0$) since there is no conduction delay.

Returning to the analysis, the inverse transform of $\bar{\theta}$ in equation (4) is obtained using contour integration with branch points at $p = 0$ and $p = -1/\varepsilon$. The contour and other details are analogous to those for a semi-infinite slab with an imposed step change in surface temperature [12]. Thus, the inverted temperature $\theta(\eta, \xi)$ is

$$\theta = (1 - I/\pi)H(\xi - \eta\varepsilon^{1/2}), \tag{7}$$

where:

$$I = \int_0^{1/\varepsilon} \frac{e^{-\xi u}}{u} \times \left[\frac{(1-\varepsilon u)\sin\{\eta[u(1-\varepsilon u)]^{1/2}\} + [u(1-\varepsilon u)]^{1/2}\cos\{\eta[u(1-\varepsilon u)]^{1/2}\}}{u+(1-\varepsilon u)}\right] du. \tag{7a}$$

Equations (7) and (7a) reduce to Fourier expressions for $\varepsilon = 0$. Further, the transformation $z = 1/u$ removes the singularity at $u = 0$ in equation (7a), giving the equivalent expression

$$I = \int_{\varepsilon}^{\infty} \frac{e^{-\xi/z}}{z} \left[\frac{(z-\varepsilon)\sin\{(\eta/z)(z-\varepsilon)^{1/2}\} + (z-\varepsilon)^{1/2}\cos\{(\eta/z)(z-\varepsilon)^{1/2}\}}{1+(z-\varepsilon)} \right] dz. \quad (7b)$$

Finally, setting $\eta = 0$ in equations (7)–(7b) gives expressions for transient surface temperature.

3. ILLUSTRATION OF RESULTS

Slab heating

The results for heating of the slab are illustrated here for a turbulent flow of air at atmospheric pressure and velocity of 35 m s^{-1}. A representative value [13] for h is 78.07 W (m^{-2} C^{-1}) using $T_\infty = 60$°C and $T_0 = 20$°C. Also, $\varepsilon = 0.02$ using values for processed meat [1] : $\tau = 15$ s, $k = 0.80$ W (m C)$^{-1}$ and $\alpha = 1.40 \times 10^{-7}$ m^2 s^{-1}.

Equations (6), (7) and (7b) give the predictions for the hyperbolic case. The numerical integrations needed for equation (7b) are performed with Simpson's rule using an upper limit of 1×10^5 to approximate the upper limit of infinity. With this approximate limit, and sufficient subdivisions of the integration range, there is less than a 1% change in predicted temperatures when the limit is increased. These predictions are compared to those for the Fourier case ($\varepsilon = 0$) obtained with its expression for temperature ([14], or equations (7) and (7a)).

The key feature of this comparison is the higher temperature of the hyperbolic case. This higher temperature results from the conduction delay that, at first, confines heating of the slab by the thermal wave to a thin internal region adjacent to the surface. Thus, the step increase in surface temperature given by equation (6) reflects the initial conduction delay. Further, the equation shows the step change becomes larger as ε (hence τ) increases, since the delay increases.

In contrast, temperatures for the Fourier case are lower since heat is immediately conducted into the slab. Consequently, surface temperature for the Fourier case does not show a step increase. As time increases, however, the hyperbolic temperatures converge to the Fourier values since the wave decays while propagating into the slab.

More specifically, Fig. 1 compares surface temperature vs time for the hyperbolic (solid line) and Fourier (dashed line) cases. The maximum difference between cases occurs as heating begins at $\xi = 0^+$, when $\theta_{step} = 0.12$ for the hyperbolic case. In comparison, the Fourier temperature initially remains at $\theta = 0$.

The maximum difference in surface temperature is about 25%, relative to the Fourier value. In practical terms, this difference may not be important since there is a "rule of thumb" uncertainty of $\approx$20% in empirical values for h. However, the difference between cases increases as ε becomes larger by increasing τ or h. In particular, increasing τ raises the "hyperbolic temperature" by increasing the conduction delay, while increasing h raises this temperature by increasing the amount of heat transferred, then confined, near the surface. For instance, replacing the air with engine oil of $h = 200$ W (m^{-2} C^{-1}) gives $\varepsilon = 0.13$ and $\theta_{step} = 0.27$. Now, the maximum difference in surface temperature is increased to about 55%, relative to the Fourier value $\theta = 0$ (or 20°C). This larger difference may be important, being considerably greater than the "rule of thumb" uncertainty in h.

Finally, Fig. 1 shows that the surface temperature for the hyperbolic case converges to the Fourier value as time increases. For example, the difference between cases is reduced to approximately 5% by $\xi = 0.10$ (or 5τ).

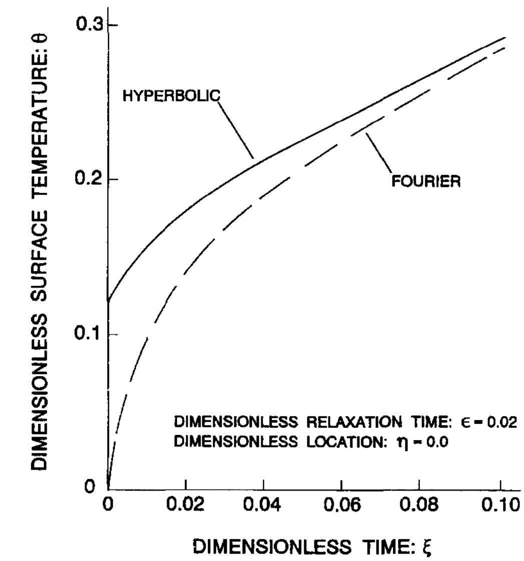

Fig. 1. Comparison of surface temperature vs time for hyperbolic and Fourier cases.

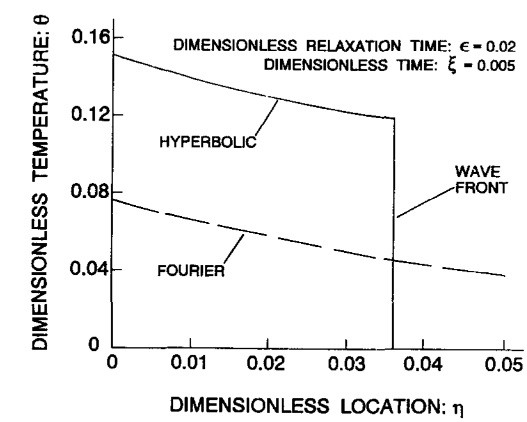

Fig. 2. Comparison of internal temperature profiles for hyperbolic and Fourier cases.

Next, Fig. 2 compares internal temperature profiles at the dimensionless time of $\xi = 0.005$ when $\eta_{wave} \approx 0.36$ using convection conditions for air cited with Fig. 1. The temperature in Fig. 2 for the hyperbolic case is again larger because of conduction delay. Further, the difference between cases would again grow larger by increasing ε. However, the maximum difference always occurs at the surface since the wave decays while propagating into the slab. Although not shown in Fig. 2, the hyperbolic prediction converges to the Fourier value as time increases.

Slab cooling

Equations (6)–(7b) also apply to cooling of the slab since the dimensionless variables are the same for cooling. However, the effect of cooling becomes evident after reverting to original variables. For example, with $T_\infty = 20$°C and $T_0 = 60$°C, the hyperbolic surface temperature has a step *decrease* to approximately 55°C when subjected to cooling with the same values of h, τ, etc. as for the air convection cited previously. This decrease occurs because of conduction delay that, initially, prevents replacing heat removed from the surface with heat from deeper in the slab. For the Fourier case, however, the surface temperature is still initially unchanged because of immediate replacement. Thus, the hyperbolic temperature is now less than the Fourier value.

In general, hyperbolic temperatures during cooling are less than Fourier values because of conduction delay. Consequently, internal temperature profiles for the hyperbolic case would show a thermal wave separating "uncooled" and "cooled" regions in the slab. These hyperbolic temperatures would converge to Fourier values as time increases.

Conditions for importance of hyperbolic conduction

Equation (6) permits quick estimates of conditions for which hyperbolic conduction is important since θ_{step} involves the maximum difference between hyperbolic and Fourier temperatures, as described previously. Specifically, $\theta_{\text{step}} = (T_{\text{step}} - T_0)/(T_\infty - T_0)$ is the maximum difference that actually occurs between hyperbolic and Fourier surface temperatures divided by the largest difference that could occur. Hence, setting a minimum value of θ_{step} as a criterion for importance of hyperbolic conduction, then asking "What values of $1/(1+\varepsilon^{-1/2})$ cause this criterion to be met or exceeded?" leads to

$$\frac{\tau\alpha h^2}{k^2} \geqslant \frac{\theta_{\text{step}}^2}{(1-\theta_{\text{step}})^2}. \tag{8}$$

Thus, equation (8) gives values of τ and h etc., for which hyperbolic conduction is important. For example, setting $\theta_{\text{step}} = 0.2$ as the criterion and using values for τ, α and k cited previously for processed meat shows $h \geqslant 138.0$ W/(m^2 °C) for hyperbolic conduction to be important.

Agreement with numerical solution

The analytical solution obtained here for the semi-infinite slab agrees with a numerical solution [15] (e.g. finite difference) for a finite slab. In the numerical solution, one surface of the finite slab is subjected to a pulsed heat flux while the other surface is heated by convection. When viewed from either surface, this slab behaves as a semi-infinite slab before the thermal waves originating at both surfaces meet inside the slab. Thus, before the waves meet, the temperature profile given in ref. [15] for convection is the same as that obtained with equations (7) and (7a) or (7b).

4. CONCLUSION

In conclusion, temperatures predicted with the hyperbolic equation can be significantly different from those of the Fourier equation for a slab with surface convection. For heating of the slab, hyperbolic temperatures are initially greater than Fourier values. For cooling, however, the temperatures are initially less than Fourier values. For both heating and cooling, hyperbolic temperatures converge to Fourier values as times increases.

These differences have important implications since the hyperbolic equation may be valid for some materials, possibly including human tissue. For example, burn damage to tissue increases rapidly with increasing temperature. Thus, damage predictions currently based on Fourier's law would be more severe using the hyperbolic equation. Further, "cooling protocols" developed with Fourier's law for cryopreserving human organs may need to be re-examined because organ damage is sensitive to temperature. Here, the lower temperatures that would be predicted with the hyperbolic equation suggest an increased risk of damage not anticipated using Fourier's law.

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